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On the Geometry of Orthomodular Spaces over Fields of Power Series

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Orthomodular spaces are generalizations of Hilbert spaces with which they share the basic property expressed by the Projection Theorem. We study two infinite-dimensional orthomodular spaces, both constructed over the same field of power series, but with different inner products. On the first space every bounded, self-adjoint operator decomposes into an orthogonal sum of operators of rank 1 or 2; on the second space, in contrast, there exist self-adjoint operators that are undecomposable. These differences reflect the fact that the underlying geometries are dissimiliar.

1. INTRODUCTION

A hermitian space (E, Φ) (over any field) is called orthomodular if the lattice $\mathcal{L}(E, \Phi)$ of all orthogonally closed subspaces satisfies the orthomodular law. Besides the classical examples of real, complex, or quaternionic Hilbert spaces, there are nonclassical orthomodular spaces of infinite dimension which are constructed over certain non-archimedianly valued complete fields and are endowed with a natural non-archimedian norm. We consider the algebra $\mathcal{B}(E)$ of all bounded linear operators on such orthomodular spaces.

The main problem is whether a given self-adjoint, bounded operator can be decomposed orthogonally. The answer is found to depend on both the arithmetic of the base field and the geometry of the space. In Keller and Ochsenius (1994, 1995) we stessed the role of the arithmetic; in the present paper we concentrate on the geometry.

We shall deal exclusively with spaces over a field $K = \mathbf{R}((\Gamma))$ of generalized power series with real coefficients and exponents in a group Γ.

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K is henselian, so the influence of the arithmetic is minimalized. We compare two orthomodular spaces, $(E^{(1)}, \Phi^{(1)})$ and $(E^{(2)}, \Phi^{(2)})$, which are both constructed over K by the same general procedure, but with different inner products. The two spaces have the same topological and analytical features, but their geometries diverge. The geometric differences have a strong bearing on operators. In fact, every self-adjoint operator on $E^{(1)}$ gives rise to a representation of E as the closure of an orthogonal sum of invariant subspaces of dimensions 1 or 2. On the space $E^{(2)}$, in turn, there exist bounded, self-adjoint operators which do not admit any invariant closed subspace at all.

In Section 2 we describe the base field $K = \mathbf{R}((\Gamma))$ and the method of constructing orthomodular spaces over K. In Section 3 we present the results on operators. The paper is expository. We give outlines of proofs when they cast light on the underlying geometric ideas.

2. CONSTRUCTION OF ORTHOMODULAR SPACES

2.1 The Base Field

We start with a direct sum

$$\Gamma := \mathbf{Z} \oplus \mathbf{Z} \oplus \cdots \oplus \mathbf{Z} \oplus \cdots$$

of countably many copies of the group of integers. Γ is an abelian, additive group under componentwise operations. We order Γ antilexicographically.

Let $K := \mathbf{R}((\Gamma))$ be the field of all generalized power series with exponents in Γ and coefficients in $\mathbf{R}$. Thus K consists of all series

$$\xi = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma \qquad (a_\gamma \in \mathbf{R})$$

for which the support

$$\text{supp}(\xi) := \{\gamma \in \Gamma | a_\gamma \neq 0\}$$

is a well-ordered subset of Γ. The operations on K are the obvious ones: if $\xi = \Sigma_{\gamma\in\Gamma} a_\gamma t^\gamma$ and $\eta = \Sigma_{\gamma\in\Gamma} b_\gamma t^\gamma$, then $\xi + \eta = \Sigma_{\gamma\in\Gamma}(a_\gamma + b_\gamma)t^\gamma$ and $\xi \cdot \eta = \Sigma_{\gamma\in\Gamma} c_\gamma t^\gamma$, where $c_\gamma := \Sigma_{\delta+\delta'=\gamma} a_\delta b_{\delta'}$.

There is a natural Krull valuation $v: K \to \Gamma \cup \{\infty\}$ on K defined by

$$v(\xi) := \min \text{supp}(\xi) \quad \text{if} \quad \xi \neq 0; \qquad v(\xi) = \infty \quad \text{if} \quad \xi = 0$$

The valued field (K, v) is a complete and henselian (Ribenboim, 1950).

2.2. The Space (E, Φ)

There is a general method which allows us to construct a host of orthomodular spaces over K. The procedure involves a technical device called "types" (for details we refer to Gross and Künzi, 1980, p. 199). The value group of (K, v) is $\Gamma = \mathbf{Z} \oplus \mathbf{Z} \oplus. \ldots$ The elements of the quotient group $\Gamma/2\Gamma \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \ldots$ are called the (algebraic) types of the field (K, v). The type of $\gamma \in \Gamma$, denoted by $T(\gamma)$, is simply its image in $\Gamma/2\Gamma$. The type of a scalar $0 \neq \lambda \in K$ is defined by $T(\lambda) := T(v(\lambda))$. Then $T(\xi^2\lambda) = T(\lambda)$ for $0 \neq \xi \in K$.

Now we choose a sequence $\Lambda := (\lambda_i)_{i \in N_0}$ of scalars $\lambda_i \in K$ for which

$$T(\lambda_i) \neq T(\lambda_j) \qquad \text{for all} \quad i \neq j \tag{*}$$

From (*) we deduce that $(\lambda_i)_{i \in \mathbf{N}_0}$ satisfies the so-called type condition: if, for any sequence $(\xi_i)_{i \in \mathbf{N}_0}$, the set $\{v(\lambda_i \xi_i^2) \mid i \in \mathbf{N}_0\} \subset \Gamma$ is bounded from below, then $\lambda_i \xi_i^2 \to 0$ as $i \to \infty$.

Consider

$$E := \left\{ (\xi_i)_{i \in \mathbf{N}_0} \in K^{\mathbf{N}_0} \,\middle|\, \text{the series } \sum_{i=0}^{\infty} \xi_i^2 \lambda_i \text{ converges in the valuation topology} \right\}$$

Notice that a series $\Sigma_{i=0}^{\infty} \xi_i^2 \lambda_i$ converges iff $\xi_i^2 \lambda_i \to 0$. It follows that E is a vector space under componentwise operations. We define a symmetric, bilinear form Φ on E by

$$\Phi(x, y) := \sum_{i=0}^{\infty} \xi_i \eta_i \lambda_i \qquad \text{for} \quad x = (\xi_i)_{i \in \mathbf{N}_0}, y = (\eta_i)_{i \in \mathbf{N}_0}$$

We list the basic properties of the above space.

Theorem 1. (Gross and Küenzi, 1980, Theorem 28):

(i) (E, Φ) is an orthomodular space.

(ii) The map $\|\cdot\| : E \to \Gamma \cup \{\infty\}$ defined by $\|x\| := v(x, x)$ is a non-archimedian norm on E. The form Φ is continuous in the norm topology. E is complete in the norm topology.

(iii) A linear subspace U of E is closed in the norm topology if and only if it is orthogonally closed.

(iv) If $x, y \in E$ are orthogonal, $x \perp y$, then $\|x\| \neq \|y\|$. Consequently (E, Φ) is not isometric to any proper subspace

Properties (i)–(iii) show the close analogy to classical Hilbert spaces, while (iv) gives evidence of remarkable new geometric features of the orthomodular spaces (E, Φ).

2.3. The Standard Base

For $i \in \mathbf{N}_0$ we let

$$e_i := (0, \ldots, 0, 1, 0, \ldots) \in E$$

be the vector that has 1 in place $(i + 1)$ and 0 in all other places. Then $\Phi(e_i, e_j) = 0$ for $i \neq j$ and $\Phi(e_i, e_i) = \lambda_i$. Now, $\{e_i | i \in \mathbf{N}_0\}$ is an orthogonal continuous base of (E, Φ), that is, every vector $x \in E$ can be expressed as

$$x = \sum_{i=0}^{\infty} \xi_i e_i = \lim_{n\to\infty} \left(\sum_{i=0}^{n} \xi_i e_i \right)$$

We observe that, by Theorem 1(iv), the base $\{e_i | i \in \mathbf{N}_0\}$ cannot be normalized (Solèr, 1995).

2.4. Residual Spaces

For $n = 0, 1, 2, \ldots$ the set

$$\Delta_n := \underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{n \text{ times}} \oplus \{0\} \oplus \{0\} \oplus \cdots \subset \Gamma$$

is a convex (or isolated) subgroup of Γ. To each Δ_n there corresponds, by general valuation theory, a valuation ring $R_n := \{\xi \in K | \varphi(\xi) \geq \delta$ for some $\delta \in \Delta_n\}$ with maximal ideal $J_n := \{\xi \in K | \varphi(\xi) > \delta$ for all $\delta \in \Delta_n\}$. The quotient $\hat{K}_n := R_n/J_n$ is called the residue field belonging to Δ_n. $\hat{K}_n$ is isomorphic to $\mathbf{R}((\Delta_n))$, the field of generalized power series with exponents in Δ_n (Ribenboim, 1950).

Next, the sets

$$M_n := \{x \in E | \Phi(x, x) \in R_n\}, \qquad S_n := \{x \in E | \Phi(x, x) \in J_n\}$$

are R_n-submodules of E. The quotient $\hat{E}_n := M_n/S_n$ is naturally a vector space over $\hat{K}_n$. Moreover, the form Φ induces a symmetric bilinear form $\hat{\Phi}_n$ on $\hat{E}_n$. We call $(\hat{E}_n, \hat{\Phi}_n)$ the residual space of (E, Φ) belonging to the convex group Δ_n.

Let π_n: $M_n \to \hat{E}_n = M_n/S_n$ be the canonical epimorphism. Every linear subspace U of E is reduced under π_n to a linear subspace $\hat{U}_n = \pi_n(U) := \{\pi_n(x) | x \in U \cap M_n\}$. The reduction map π_n preserves orthogonality, i.e., if $U \perp W$, then $\pi_n(U) \perp \pi_n(W)$.

2.5. Bounded Linear Operators

A linear operator B: $E \to E$ is called bounded if the subset

$$\{\|B(x)\| - \|x\| \mid 0 \neq x \in E\} \subset \Gamma$$

has a lower bound in Γ. Clearly a bounded operator is continuous with respect to the norm topology. However, in contrast to the classical case of Hilbert spaces, there exist linear operators on (E, Φ) which are continuous, but not bounded. Notice also that a bounded operator cannot be assigned a norm in the usual way because a bounded subset of Γ may fail to have a supremum.

3. DECOMPOSITION OF LINEAR OPERATORS

In this section we examine two particular orthomodular spaces, $(E^{(1)}, \Phi^{(1)})$ and $(E^{(2)}, \Phi^{(2)})$, and we study the problem of orthogonal decompositions of bounded, self-adjoint operators. The spaces are obtained by specifying the sequence $\Lambda = (\lambda_i)_{i \in \mathbf{N}_0}$ on which the construction of Section 2.2 is based.

For $i = 1, 2, \ldots$, we let

$$\gamma_i := (0, \ldots, 0, 1, 0, 0, \ldots) \in \Gamma \qquad \text{where 1 is in the } i\text{th place}$$

and we let $t_i \in K = \mathbf{R}((\Gamma))$ be the one-term series $t_i := 1 \cdot t^{\gamma_i}$. Thus $v(t_i) = \gamma_i$ and, in particular, $T(t_i) \neq T(t_j)$ for $i \neq j$. It follows that the elements t_i $(i \in \mathbf{N})$ are algebraically independent over $\mathbf{R}$.

Let $n \in \mathbf{N}$. The residue field $\hat{K}_n$ is isomorphic to $\mathbf{R}((\Delta_n))$. Every $\delta \in \Delta_n$ can be written as $\delta = r_1\gamma_1 + \ldots + r_n\gamma_n$ for some $r_1, \ldots, r_n \in \mathbf{Z}$. Then $t^\delta = t_1^{r_1} \cdot \ldots \cdot t_n^{r_n}$. From this remark we derive the following description $\hat{K}_n$.

Lemma 2. The residue field $\hat{K}_n$ is isomorphic to the field $\mathbf{R}((t_1, \ldots, t_n))$ of all formal power series in $t_1, \ldots, t_n$.

Notice that $\hat{K}_n$ is isomorphic to a subfield of K, namely the closure of the subfield generated by $\{t_1, \ldots, t_n\}$ over $\mathbf{R}$.

3.1. A First Example

We put

$$\lambda_0^{(1)} = t_0 := 1, \qquad \lambda_i^{(1)} := t_i \qquad \text{for} \quad i = 1, 2, \ldots$$

Clearly the sequence $(\lambda_i^{(1)})_{i \in \mathbf{N}_0}$ satisfies the condition (*), so we can apply the construction of Section 2.2. We let $(E^{(1)}, \Phi^{(1)})$ be the orthomodular space thus obtained.

The residual spaces of $(E^{(1)}, \Phi^{(1)})$ are easily described. Indeed, since $\Phi^{(1)}(e_i, e_i) = t_i$, it follows that $\pi_n(e_i) \neq 0$ for $i \leq n$ and $\pi_n(e_i) = 0$ for $i > n$. Noticing that π_n is continuous and preserves orthogonality, we see that $\hat{E}_n^{(1)} = \pi_n(E)$ is spanned by the vectors $\hat{e}_0 = \pi_n(e_0), \ldots, \hat{e}_n := \pi_n(e_n)$. Moreover,

$$\hat{\Phi}_n^{(1)}(\hat{e}_i, \hat{e}_j) = 0 \quad \text{for} \quad 0 \leq i < j \leq n,$$

$$\hat{\Phi}_n^{(1)}(\hat{e}_i, \hat{e}_i) = t_i \quad \text{for} \quad 0 \leq i \leq n$$

We have shown:

Lemma 3. Let $(\hat{E}_n^{(1)}, \hat{\Phi}_n^{(1)})$ be the residual space of $(E^{(1)}, \Phi^{(1)})$ belonging to Δ_n. Then:

(i) $\dim \hat{E}_n^{(1)} = n + 1$.

(ii) $\hat{\Phi}_n^{(1)} \simeq \operatorname{diag}(1, t_1, t_2, \ldots, t_n)$.

Now we look at operators. The main result is the following.

Theorem 4. Every bounded, self-adjoint linear operator A: $E^{(1)} \to E^{(1)}$ gives rise to a representation of $E^{(1)}$ as the closure of an orthogonal direct sum of countably many invariant subspaces each of which is of dimension 1 or 2. That is, A can be represented as $A = \Sigma_{i=0}^{\infty} Q_i$, where the Q_i are pairwise orthogonal operators of rank 1 or 2.

For a proof we refer to Keller and Ochsenius (1995b).

3.2. A Second Example

We express every integer $i \in \mathbf{N}$ as a dual number,

$$i = \varepsilon_0 \cdot 2^0 + \varepsilon_1 \cdot 2^1 + \varepsilon_2 \cdot 2^2 + \cdots + \varepsilon_{k-1} \cdot 2^{k-1}$$

where $k \in \mathbf{N}$ and $\varepsilon_0, \ldots, \varepsilon_{k-1} \in \{0, 1\}$, and we put

$$\lambda_i^{(2)} := t_1^{\varepsilon_0} t_2^{\varepsilon_1} t_3^{\varepsilon_2} \cdots t_k^{\varepsilon_{k-1}}$$

We have

$$\begin{aligned} v(\lambda_i^{(2)}) &= v(t_1^{\varepsilon_0} t_2^{\varepsilon_1} t_3^{\varepsilon_2} \cdots t_k^{\varepsilon_{k-1}}) \\ &= \varepsilon_0 \gamma_1 + \varepsilon_1 \gamma_2 + \cdots + \varepsilon_{m-1} \gamma_m \\ &= (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m-1}, 0, \ldots) \end{aligned}$$

from which we easily deduce that the sequence $\Lambda^{(2)} := (\lambda_i^{(2)})_{i \in \mathbf{N}_0}$ satisfies the requirement (*). Applying the construction of Section 2.2, we obtain an orthomodular space $(E^{(2)}, \Phi^{(2)})$.

We look at the residual spaces.

Lemma 5. Let $(\hat{E}_n^{(2)}, \hat{\phi}_n^{(2)})$ be the residual space of $(E^{(2)}, \Phi^{(2)})$ belonging to Δ_n.

(i) $\{\pi_n(e_0), \pi_n(e_1), \ldots, \pi_n(e_{2^n - 1})\}$ is an orthogonal base of $\hat{E}_n^{(2)}$. Thus $\dim \hat{E}_n^{(2)} = 2^n$.

(ii) $\hat{\Phi}_n^{(2)} \simeq \operatorname{diag}(1, t_1, t_2, t_1 t_2, \ldots, t_1 t_2, \ldots 1_n) \simeq \operatorname{diag}(1, t_1) \otimes \ldots \otimes \operatorname{diag}(1, t_n)$.

We now state the main result.

Theorem 6. On the orthomodular space $(E^{(2)}, \Phi^{(2)})$ there exist bounded, self-adjoint operators which do not admit any proper, closed invariant subspace.

Outline of Poof. (1) We define recursively matrices $\mathcal{A}_1, \mathcal{A}_2, \ldots$ as follows. For $n \in \mathbf{N}$ we let $\mathcal{I}_{n-1}$ be the unit matrix of size $2^{n-1} \times 2^{n-1}$. We put

$$\mathcal{A}_1 := \begin{bmatrix} 0 & t_1 \\ 1 & 0 \end{bmatrix}; \qquad \mathcal{A}_n := \begin{bmatrix} \mathcal{A}_{n-1} & t_n \cdot \mathcal{I}_{n-1} \\ \mathcal{I}_{n-1} & \mathcal{A}_{n-1} \end{bmatrix}$$

Then $\mathcal{A}_n$ is a matrix of size $2^n \times 2^n$ with entries in $\mathbf{R}((t_1, \ldots, t_n))$.

For each $n \in \mathbf{N}$, the matrix $\mathcal{A}_n$ defines a linear operator A_n on the residual space $\hat{E}_n^{(2)}$ (with respect to the canonical base $\{\pi_n(e_0), \ldots, \pi_n(e_{2^n-1})\}$). It is readily verified that A_n is self-adjoint. In Keller and Ochsenius (n.d.) we establish the following basic result.

Lemma 7. The operators A_n are orthogonally undecomposable.

(2) We now prove that the sequence of operators $(A_n)_{n\in\mathbf{N}}$ on the residual spaces can be lifted to the whole space E, i.e., there exists an operator $\bar{A}$: $E \to E$ such that A_n is equal to the operator $\bar{A}_n$ induced by A on $\hat{E}_n^{(2)}$. Let $x \in E$. Then $\pi_n(x)$ is defined for all n from some n_0 on. For every $n \geq n_0$ we choose a vector $y_n \in E$ such that $\pi_n(y_n) = A_n(\pi_n(x))$. From the definition of the matrices $\mathcal{A}_n$ we deduce that $(y_n)_{n\geq n_0}$ is a Cauchy sequence. We define A: $E \to E$ by putting $A(x) := \lim_{n\to\infty} y_n \in E$. A is well-defined, A is self-adjoint, and $\hat{A}_n = A_n$ by construction.

3. Suppose, indirectly, that E admits a nontrivial closed subspaces U which is invariant under A. Then $E = U \oplus U^\perp$ by orthomodularity. Pick $0 \neq u \in U$, $0 \neq v \in U^\perp$, and $n \in \mathbf{N}$ such that $\pi_n(u) \neq 0 \neq \pi_n(v)$. Then $\hat{E}_n^{(2)} = \pi_n(E) = \pi_n(U) \oplus \pi_n(U^\perp)$ is a nontrivial decomposition of the residual space $\hat{E}_n^{(2)}$. Now U is invariant under A, so $\pi_n(U)$ is invariant under the reduced operator $\hat{A}_n$, which is equal to A_n. This contradicts Lemma 7 and the proof is complete.

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